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# Interest Rate Models

## MA3418 Mini-project

# The term structure of interest rates

- The term structure of interest rates is the relationship between interest rates or bond yields and different terms or maturities.
- Zero-coupon bond price $B(t,T)$ is the price at t for £1 payable at T.
- Spot-rate curve $R(t,T)$ is the constant interest rate applicable over the period from t to T.

# The term structure of interest rates

- The Zero-coupon bond price $B(t,T)$ is related to the spot-rate curve $R(t,T)$ as:

$$R(t,T) = -\frac{1}{T-t}\log\big(B(t,T)\big) \quad for\ t < T$$

or

$$B(t,T) = e^{-R(t,T)(T-t)}$$

# Classification of interest models

- One-Factor models
  - ✓Vasicek model
  - ✓Hull & White model
  - ✓The Cox-Ingersoll-Ross model

- Multi-factor models

# Basic characteristics of Interest models

- The model should be arbitrage free (risk-neutral approach):

$$B(t,T) = \tilde{E}\left[e^{-\int_t^T R(s)ds} \mid F(t)\right] \ for \ t < T$$

- Interest rates should be positive.
- $R(t)$ or other interest rates should exhibit some form of mean-reverting behavior.

# Classification of interest models

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# Vasicek model

- The vasicek model has the stochastic differential equation as:

$$dr(t) = \alpha\big(\mu - r(t)\big)dt + \sigma d\widetilde{W}(t),$$

Where $\widetilde{W}(t)$ is a standard Brownian motion under the risk-neutral measure.

*Vasicek, O. (1977). "An equilibrium characterization of the term structure". Journal of Financial Economics. **5** (2): 177–188. doi:10.1016/0304-405X(77)90016-2*

# Vasicek model

- The zero-coupon bond price is equal to:

$$B(t,T) = e^{a(T-t)-b(T-t)r(t)}$$

Where,

$$b(\tau) = \frac{1 - e^{-\alpha\tau}}{\alpha}$$

$$a(\tau) = (b(\tau) - \tau)\left(\mu - \frac{\sigma^2}{2\alpha^2}\right) - \frac{\sigma^2}{4\alpha} b(\tau)^2 \qquad \tau = T - t$$

# The Hull & White model

- The Hull & White model has the stochastic differential equation as:

$$dr(t) = \alpha\big(\mu(t) - r(t)\big)dt + \sigma d\widetilde{W}(t)$$

or

$$dr(t) = \big(a(t) - b(t)r(t)\big)dt + \sigma d\widetilde{W}(t)$$

Hull, J. and White, A.(1990) *"Pricing interest-rate derivative securities"*, The Review of Financial Studies, Vol 3, No. 4 (1990) pp. 573–592



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# The Hull & White model

- The zero-coupon bond price is equal to:

$$B(t,T) = e^{-a(T-t)-c(T-t)r(t)}$$

Where,

$$c(\tau) = \int_t^T e^{-\int_t^s b(v)dv} ds$$

$$a(\tau) = \int_t^T \left( a(s)c(T-s) - \frac{1}{2}\sigma^2 c^2(T-s) \right) ds \qquad \tau = T - t$$



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# THE CIR model

- The CIR model has the stochastic differential equation as:

$$dr(t) = \alpha\big(\mu - r(t)\big)dt + \sigma\sqrt{r(t)}\,d\widetilde{W}(t)$$

Cox, J.C., J.E. Ingersoll and S.A. Ross (1985). *"A Theory of the Term Structure of Interest Rates".* Econometrica. **53**: 385–407. doi:10.2307/1911242

# THE CIR model

- The Bond price is equal to:

$$B(t,T) = e^{a(T-t)-b(T-t)r(t)}$$

Where,

$$b(\tau) = \frac{2(e^{\theta\tau} - 1)}{(\theta + \alpha)(e^{\theta\tau} - 1) + 2\theta}$$

$$a(\tau) = \frac{2\alpha\mu}{\sigma^2} log\left(\frac{2\theta e^{(\theta+\alpha)\tau/2}}{(\theta + \alpha)(e^{\theta\tau} - 1) + 2\theta}\right)$$

$$\theta = \sqrt{\alpha^2 + 2\sigma^2} \qquad \tau = T - t$$

# Further discussion

- How easy is it to calculate?
- Does the model produce realistic dynamics?
- Does the model fit historical interest-rate data adequately?
- Can the model be calibrated easily to current market data?

And Why?

# Limitation of one-factor models

- Time homogeneous or not?

  Does it suitable for long run of historical data?

- Effectively work with derivative contracts or not?

  Is the model flexible enough to cope properly with a range of derivative contracts?

And How to improve?

# References:


- Carins, A. (2004) Interest rate models: an introduction, Princeton University Press.
- Baxter, M. and Rennie, A. (1996) Financial calculus: an introduction to derivative pricin, Cambridge University Press (online resource available)
- Panjer, H. H. (2001) Financial economics: with applications to investments, insurance, and pensions, The Actuarial Foundation.

# Multi-factor model

- What about a 2-factor Vasicek model?

$$dr(t) = \alpha_r\big(m(t) - r(t)\big)dt + \sigma_{r1}d\widetilde{W}_1(t) + \sigma_{r2}d\widetilde{W}_2(t)$$

$$dm(t) = \alpha_m\big(\mu - m(t)\big)dt + \sigma_{m1}d\widetilde{W}_1(t)$$

# Requirement

- You are required to write an essay of about 15 pages in length (excluding references and appendices).
- The essay should contain both an abstract and a main body.
- Full references and other sources of information used should be included.
- You may use Appendices to include detailed calculations, data, supporting information etc.

# Requirement

- You should pick **one model** from the Vasicek, Cox-Ingersoll-Ross and Hull-White models.
- Describe the desirable characteristics of the model(s) chosen.
- Describe how the model(s) can be applied to the pricing of zero-coupon bonds and interest-rate derivatives.
- Discuss the limitations of the model(s) and show how these issues can be addressed.

# To achieve a higher mark

- Discuss how the model(s) for the term structure can be applied to insurance/actuarial science.

# Deadline

Mini-project :

Weighting: 30%

Deadline for submission: 4pm on Monday, 27 April 2020 (both hard copy and TurnItIn)

# Questions?