Since

$$\int_\rho^R f_1(x)\,dx - \int_\rho^R f_2(x)\,dx = \int_\rho^R \frac{1}{x+1}\,[e^{-a \operatorname{Log} x} - e^{-a(\operatorname{Log} x + 2\pi i)}]\,dx$$

$$= \int_\rho^R \frac{x^{-a}}{x+1}(1 - e^{-2\pi a i})\,dx,$$

we thus arrive at the result

$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \int_\rho^R \frac{x^{-a}}{x+1}\,dx = \frac{2\pi i \exp(-a\pi i)}{1 - \exp(-2a\pi i)}.$$

That is,

$$\int_0^\infty \frac{x^{-a}}{x+1}\,dx = \frac{\pi}{\sin a\pi} \qquad (0 < a < 1).$$