Thus

(3) $$\int_{C_2} f_2(z)\,dz = 2\pi i \exp(-a\pi i).$$

Since $f_1(z) = f_2(z)$ on the ray $\arg z = \phi$, it is also true that

(4) $$\int_{C_1} f_1(z)\,dz + \int_{C_2} f_2(z)\,dz = \int_\rho^R f_1(x)\,dx - \int_\rho^R f_2(x)\,dx$$
$$+ \int_{\Gamma_1} f_1(z)\,dz + \int_{\Gamma_2} f_2(z)\,dz + \int_{\gamma_1} f_1(z)\,dz + \int_{\gamma_2} f_2(z)\,dz$$

where Γ_k is the large circular arc and γ_k is the small circular arc of the simple closed contour C_k $(k = 1, 2)$ shown in Fig. 52.

When z is on Γ_k $(k = 1, 2)$,

$$|f_k(z)| = \left|\frac{z^{-a}}{z+1}\right| \leq \frac{R^{-a}}{R-1};$$

and since the arc Γ_k is a portion of a circle whose circumference is $2\pi R$,

$$\left|\int_{\Gamma_k} f_k(z)\,dz\right| \leq \frac{R^{-a}}{R-1}\,2\pi R.$$

Hence

(5) $$\lim_{R\to\infty} \int_{\Gamma_k} f_k(z)\,dz = 0 \qquad (k = 1, 2).$$

When z is on γ_k $(k = 1, 2)$,

$$|f_k(z)| = \left|\frac{z^{-a}}{z+1}\right| \leq \frac{\rho^{-a}}{1-\rho}.$$

Consequently,

$$\left|\int_{\gamma_k} f_k(z)\,dz\right| \leq \frac{\rho^{-a}}{1-\rho}\,2\pi\rho,$$

and

(6) $$\lim_{\rho\to 0} \int_{\gamma_k} f_k(z)\,dz = 0 \qquad (k = 1, 2).$$

It follows from equation (4) and the results obtained in equations (5) and (6) as well as equations (2) and (3) that

$$\lim_{\substack{R\to\infty\\ \rho\to 0}} \left(\int_\rho^R f_1(x)\,dx - \int_\rho^R f_2(x)\,dx\right) = 2\pi i \exp(-a\pi i).$$