which is important in the study of the gamma function.[1] The integral exists when $0 < a < 1$ because the integrand behaves like x^{-a} near $x = 0$ and like x^{-a-1} as x tends to infinity.

To evaluate integral (1), we consider the two line integrals

$$\int_{C_1} f_1(z)\,dz, \qquad \int_{C_2} f_2(z)\,dz$$

where

$$f_1(z) = \frac{z^{-a}}{z+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right),$$

$$f_2(z) = \frac{z^{-a}}{z+1} \qquad \left(|z| > 0, \frac{\pi}{2} < \arg z < \frac{5\pi}{2}\right)$$

and C_1 and C_2 are the simple closed contours shown in Fig. 52. In that figure $\rho < 1 < R$, and the angle ϕ is chosen so that $\pi/2 < \phi < \pi$.

Observe that the function f_1 is analytic within and on C_1; hence

$$\int_{C_1} f_1(z)\,dz = 0. \tag{2}$$

Moreover, the function f_2 is analytic within and on C_2 except for the simple pole at the point $z = -1$ which is interior to C_2. Now in the definition of f_2,

$$z^{-a} = \exp\left[-a(\text{Log}\,|z| + i \arg z)\right] \qquad \text{where} \qquad \frac{\pi}{2} < \arg z < \frac{5\pi}{2},$$

and the residue of f_2 at $z = -1$ is

$$\lim_{z \to -1} (z+1) f_2(z) = \lim_{z \to -1} z^{-a} = \exp(-a\pi i).$$

[1] See, for example, p. 4 of the book by Lebedev cited in Appendix 1.

f_1 f_2

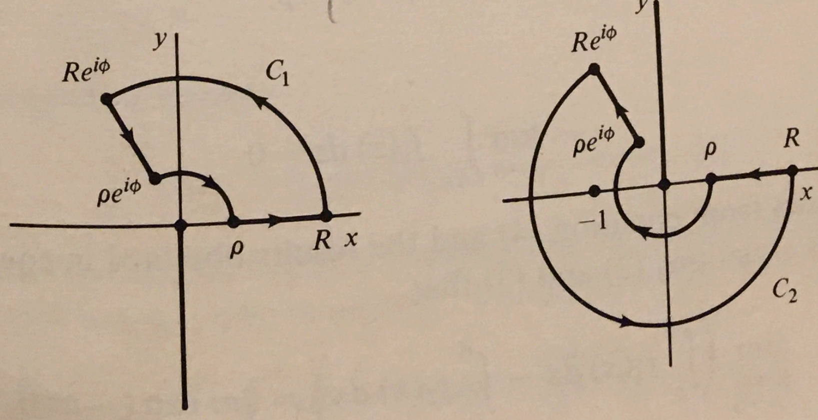

FIGURE 52